Sign of the Interference Particle Parameter in Conversion-Electron Directional Correlation*

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It is shown that the interference particle parameter, $b_\nu(ML,EL')$, appearing in the directional correlation of internal-conversion electrons, has the opposite sign from that given in the literature. It is also shown that in the high-energy limit, the directional-correlation pattern for conversion electrons is identical with that of the corresponding gamma rays.

INTRODUCTION

DIRECTIONAL–CORRELATION experiments involving conversion electrons are important sources of information about the atomic nucleus. The interpretation of such experiments is based on the formulas and tables of the classic review article of Biedenharn and Rose,[1] which is referred to hereafter as BR. Their paper presents expressions which relate the results of a conversion-electron directional-correlation measurement to the corresponding directional-correlation involving the gamma ray. This relation is formally accomplished by the introduction of factors, particle parameters, b, into the expression for the gamma-ray correlation.

The present note points out that a sign change must be introduced in the expression in BR for the particle parameter corresponding to the interference between ML and $E(L+1)$ transitions. The old result is in error. In particular, the sign of $b_\nu[ML, E(L+1)]$ in Eq. (101) of BR, as well as the signs of the numerical values of this quantity in their Table IV, must be reversed. Similar sign reversals are also necessary in the corresponding conclusions of other authors.[2–5] The signs of the expressions for the particle parameters not involving the interference between multipoles are unaffected.

BR have also discussed the high-energy limits of the particle parameters for the Coulomb case and $Z=0$. They conclude that for pure multipoles the particle parameters approach $+1$, and that for mixed multipoles, the interference particle parameters approach -1 for finite Z, and $+1$ for $Z=0$. In this note it is shown that *all* particle parameters, b_ν, approach $+1$ for both $Z=0$ and finite Z, in the high-energy limit. When the signs of the interference parameters in the formalism and tables of BR are corrected, the proper high-energy limits are obtained. It is also shown that this simple limit holds for more general and more realistic potentials than the pure Coulomb.

This sign change in the relation between the gamma-ray and conversion-electron correlation has drastic consequences in the interpretation of experimental data involving mixed transitions.

CORRELATION FORMALISM

The proper sign of the interference particle parameter is demonstrated below in a number of ways, in addition to a straightforward *recalculation* of the correlation formalism using traditional techniques. Second, the simple *free-particle case* ($Z=0$) is considered in two ways: by direct application of the formalism, and immediately from the primitive form of the conversion interaction. In the high-energy limit, both of these procedures give the same result: all particle parameters approach $+1$. Third, the *high-energy Coulomb case* is considered in the same two ways, with the same result. This result for the b's parallels the fact that the conversion coefficients are independent of multipole order in the high-energy limit. The arguments for the point-Coulomb case are also shown to apply to more general potentials.

That all the particle parameters approach $+1$ and that all conversion coefficients become equal in the high-energy limit mean that the conversion-electron and gamma-ray correlation patterns become identical. This simple physical situation can be understood in the following way. In the high-energy limit, the mechanics of the conversion process are such that the major contributions arise from distances which are large compared with the wavelength of the photon. This means that the important part of the electron functions lie in the wave zone of the radiation, that the effects of the centrifugal barrier on the ejected electron may be

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[1] L. C. Biedenharn and M. E. Rose, Rev. Mod. Phys. 25, 729 (1953). This paper contains references to earlier articles.
[2] E. V. Ivash, Nuovo Cimento 9, 136 (1958). This paper gives formulas for both EL, $M(L+1)$ and ML, $E(L+1)$ mixtures for both the K and L shells.
[3] S. Devons and J. B. Goldfarb, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. XLII.
[4] A. Z. Dolginov, *Gamma-Rays*, edited by L. A. Sliv (Academy of Sciences, U.S.S.R., 1961), Chap. 6.
[5] H. Ikegami, Phys. Rev. 120, 2185 (1960). This paper includes a discussion of $E1$, $M2$ mixtures. While no formulas are given, his Table V indicates that the interference particle parameter incorrectly approaches (-1) in the high-energy limit.

neglected, and that the initial electron momentum is negligible. Under these conditions the conversion occurs at essentially macroscopic distances, and the gamma-ray and conversion-electron patterns must necessarily follow each other. Although these considerations are implicit in any derivation of the high-energy limit, they are given explicitly with reference to the Coulomb case discussed below.

Recalculation

A detailed recalculation of the particle parameters for mixed transitions, using methods similar to those of Biedenharn and Rose,[1] has been carried out. This recalculation results in a formula for the interference parameter, $b_\nu[ML, E(L+1)]$, which has the opposite sign of the standard result, Eq. (101) of BR. *There is agreement between the present derivation and that in* BR *up to and including their* Eq. (100), *but the sign difference appears between* (100) *and* (101), *and is carried through into the evaluation of their* Table IV.[6,6a]

For the specific case of the K-shell (or any $s_{1/2}$ initial state) conversion of a mixed ML and $E(L+1)$ transition, the correct result for the mixed particle parameter[7] is, in the notation of BR:

$$b_\nu[ML, E(L+1)] = (-1)\frac{\left[\frac{L(L+2)}{(2L+1)(2L+3)}\right]^{1/2} \mathrm{Re}\left[e^{i(\theta_e-\theta_m)}\left(1+\frac{L+1}{LT_m}\right)^*\left(1-\frac{L+1}{T_e}\right)\right]}{\left[\left(1+\frac{L+1}{L|T_m|^2}\right)\left(1+\frac{(L+1)(L+2)}{|T_e|^2}\right)\right]^{1/2}}, \tag{1}$$

where

$$T_m(L) = \frac{\exp(i\delta_{L+1})}{\exp(i\delta_{-L})}\left[\int_0^\infty dr\, r^2 h_L^{(1)}(f_{\kappa f}g_i + g_{\kappa f}f_i)\right]_{\kappa f=L+1} \Big/ \left[\int_0^\infty dr\, r^2 h_L^{(1)}(f_{\kappa f}g_i + g_{\kappa f}f_i)\right]_{\kappa f=-L},$$

$$T_e(L+1) = (L+1)\frac{\exp(i\delta_{L+1})}{\exp(i\delta_{-L-2})}\left[\int_0^\infty dr\, r^2\{(L+1)h_{L+1}^{(1)}(f_{\kappa f}f_i + g_{\kappa f}g_i) - (2L+3)h_L^{(1)}f_{\kappa f}g_i - h_L^{(1)}g_{\kappa f}f_i\}\right]_{\kappa f=L+1} \Big/$$
$$\left[\int_0^\infty dr\, r^2\{(L+1)h_{L+1}^{(1)}(f_{\kappa f}f_i + g_{\kappa f}g_i) + 2(L+1)h_L^{(1)}g_{\kappa f}f_i\}\right]_{\kappa f=-L-2}, \tag{2}$$

and

$$\theta_m = \text{Phase of}\left[i\int_0^\infty dr\, r^2 h_L^{(1)}(f_{\kappa f}g_i + g_{\kappa f}f_i)\right]_{\kappa f=L+1}$$
$$\theta_e = \text{Phase of}\left[i\int_0^\infty dr\, r^2\{(L+1)h_{L+1}^{(1)}(f_{\kappa f}f_i + g_{\kappa f}g_i) - (2L+3)h_L^{(1)}f_{\kappa f}g_i - h_L^{(1)}g_{\kappa f}f_i\}\right]_{\kappa f=L+1}. \tag{3}$$

The Dirac radial functions f and g, and the continuum phases δ_κ, are defined in accord with convention,[8] and $h_L^{(1)} = h_L^{(1)}(kr)$ are the spherical Hankel functions. It is worth noting that the interference parameter, b_ν, is independent of ν.

The point made here is that the expression (1) is opposite in sign to Eq. (101) of BR, and to corresponding expressions in the literature.[2–5]

Example: The Free-Electron Case

The calculation of the formalism leading to the above result requires many steps which have no simple physical interpretation. It is advantageous, therefore, to consider a particularly simple example which allows a result to be calculated in two different ways: first, from the formalism above, and second, directly from the primitive form of the conversion interaction.[9] Such a procedure is especially useful in demonstrating the sign of the result.

[6] We have spot-checked values in Table IV of BR for $M1$, $E2$ mixtures against Eq. (101) of BR, using values of the radial integrals obtained from p. 164 *et seq.* in M. E. Rose, *Internal Conversion Coefficients* (Interscience Publishers, Inc., New York, 1958). We have also spot-checked values of the pure $M1$ and $E2$ particle parameters against BR (95a) and (97). It is to be noted that, in accordance with the phase convention used by BR [see their Eq. (94)], the values of R_2 for both $M1$ and $E2$ transitions given on pp. 164 *et seq.* of this latter reference, must be multiplied by -1.

[6a] *Note added in proof.* L. Biedenharn and M. E. Rose have sent us a preprint of a recalculation of $b_\nu[ML, E(L+1)]$ which agrees with the sign correction presented here.

[7] For completeness we give also the form of the pure multipole particle parameters from BR:

$$b_\nu[ML] = 1 + \frac{\nu(\nu+1)}{2L(L+1)-\nu(\nu+1)}\frac{L(L+1)}{2L+1}\frac{|1-T_m|^2}{L+1+L|T_m|^2},$$

$$b_\nu[E(L+1)] = 1 + \frac{\nu(\nu+1)}{2(L+1)(L+2)-\nu(\nu+1)}\frac{L+1}{2L+3} \times\frac{|L+2+T_e|^2}{(L+1)(L+2)+|T_e|^2}.$$

[8] M. E. Rose, Phys. Rev. **51**, 484 (1937), and BR Eq. (94).

[9] The method used here closely resembles the treatment of a very similar problem by N. M. Kroll and W. Wada, Phys. Rev. **98**, 1355 (1955).

The example considered consists of using free-particle Dirac wave functions of momentum $\mathbf{p}$ for the final-state electron, and a corresponding free-particle initial state of zero kinetic energy. Further, we deliberately consider the situation where all penetration effects[10] are negligible. Finally, only the high-energy limit of this case is considered. Some care must be taken in defining this limit. In particular, it is understood that kR, the transition wave number times the "source" radius, remains small, as is usual in low-energy nuclear physics.

The calculation of the particle parameters for this free-particle case using the formalism (1)–(3) is straightforward.[11] The results are:

$$b_\nu(ML)=1$$
$$b_\nu[E(L+1)]=1+\frac{4\nu(\nu+1)(L+1)}{[2(L+1)(L+2)-\nu(\nu+1)][(L+2)k^2+4(L+1)]} \tag{4}$$
$$b_\nu[ML,E(L+1)]=\frac{1}{[1+4(L+1)/k^2(L+2)]^{1/2}}.$$

These results are in agreement with the $Z=0$ expressions written on pages 755 and 759 of BR. However, it is important to note that had this calculation been based on Eq. (101) of BR, the sign of the interference parameter, $b_\nu[ML,E(L+1)]$, would have been reversed. The correct sign on pages 755 and 759 of BR was obtained by their use of correct expressions previous to their final Eq. (101).

In the high-energy limit, $k\rightarrow\infty$, all of the above expressions, Eq. (4), approach $+1$. This, taken with the fact that in this limit the conversion coefficients are independent of multipole order, means that the conversion-electron and gamma-ray directional-correlation patterns become identical for any mixture of ML and $E(L+1)$ multipoles. We will now derive this simple result directly from the primitive conversion interaction.

The electromagnetic interaction between the nucleus and electron is, to lowest order in e^2 (which is all that will be considered here),

$$H'=-\int d\tau_n d\tau_e(\mathbf{j}_n\cdot\mathbf{j}_e-\rho_n\rho_e)\frac{e^{ik|\mathbf{r}_n-\mathbf{r}_e|}}{|\mathbf{r}_n-\mathbf{r}_e|}. \tag{5}$$

For the free-particle case, a plane-wave representation of this interaction permits an immediate evaluation. In the plane-wave representation

$$H'=-\frac{1}{2\pi^2}\int d\tau_n d\tau_e d\mathbf{k}'(\mathbf{j}_n\cdot\mathbf{j}_e-\rho_n\rho_e)\frac{e^{i\mathbf{k}'\cdot(\mathbf{r}_n-\mathbf{r}_e)}}{k'^2-k^2}. \tag{6}$$

This Feynman expression can now be rewritten[12]

$$H'=-\frac{1}{2\pi^2}\int d\tau_n d\tau_e d\mathbf{k}'\sum_j \mathbf{j}_n\cdot\hat{e}_j\mathbf{j}_e\cdot\hat{e}_j\frac{e^{i\mathbf{k}'\cdot(\mathbf{r}_n-\mathbf{r}_e)}}{k'^2-k^2}+\frac{1}{2\pi^2}\int d\tau_n d\tau_e d\mathbf{k}'(\mathbf{j}_n\cdot\hat{k}'\mathbf{j}_e\cdot\hat{k}')\frac{1}{k^2}e^{i\mathbf{k}'\cdot(\mathbf{r}_n-\mathbf{r}_e)}, \tag{7}$$

[10] E. L. Church and J. Weneser, Ann. Rev. Nucl. Sci. **10**, 193 (1960).

[11] We consider the specific case of the mixed conversion of ML and $E(L+1)$ transitions from a $\kappa_i=-1$ initial state. The radial Dirac wave functions, f_κ, g_κ, for the initial state are simply:

$$g_{-1}=1,\quad f_{-1}=0.$$

The corresponding functions for the final electron states of ML conversion are:

$$g_{-L}=+\left(\frac{W+1}{\pi p}\right)^{1/2}pj_{L-1}(pr),\quad f_{-L}=-\left(\frac{W-1}{\pi p}\right)^{1/2}pj_L(pr),$$
$$g_{L+1}=-\left(\frac{W+1}{\pi p}\right)^{1/2}pj_{L+1}(pr),\quad f_{L+1}=-\left(\frac{W-1}{\pi p}\right)^{1/2}pj_L(pr).$$

The functions for the final electron states of $E(L+1)$ conversion are:

$$g_{L+1}=-\left(\frac{W+1}{\pi p}\right)^{1/2}pj_{L+1}(pr),\quad f_{L+1}=-\left(\frac{W-1}{\pi p}\right)^{1/2}pj_L(pr),$$
$$g_{-L-2}=+\left(\frac{W+1}{\pi p}\right)^{1/2}pj_{L+1}(pr),\quad f_{-L-2}=-\left(\frac{W-1}{\pi p}\right)^{1/2}pj_{L+2}(pr).$$

In the above $W=(p^2+1)^{1/2}$ is equal to the total energy of the ejected electron and $W-1$ is the nuclear transition energy, k. The above final-state wave functions are in agreement with the normalization and phase conventions (Ref. 8).

Substituting the above wave functions into the previous expressions, Eqs. (2) and (3), we find for this case of $Z=0$:

$$T_m(L)=1,$$
$$T_e(L+1)=(L+1)-(2L+3)k/(k+2),$$
$$\theta_e-\theta_m=\text{Phase of }T_e(L+1).$$

In evaluating the above the general relation

$$\int_0^\infty dr\, r^2h_L^{(1)}(kr)j_L(pr)=\frac{i}{k}\left(\frac{p}{k}\right)^L\frac{1}{k^2-p^2},\quad p\neq k$$

has been used.

Substitution of the above values into Eq. (1) gives the particle parameters quoted in Eq. (4) in the text.

[12] The transformation steps are given here in full. The use of the identity

$$\frac{1}{k'^2-k^2}=\frac{k'^2}{k^2}\frac{1}{k'^2-k^2}-\frac{1}{k^2},$$

and the explicit separation of the currents into transverse and longitudinal parts by means of the identity

$$\mathbf{j}_n\cdot\mathbf{j}_e=\sum_j \mathbf{j}_n\cdot\hat{e}_j\mathbf{j}_e\cdot\hat{e}_j+\mathbf{j}_n\cdot\hat{k}'\mathbf{j}_e\cdot\hat{k}'$$

puts H' into the form

$$H'=-\frac{1}{2\pi^2}\int d\tau_n d\tau_e d\mathbf{k}'\Big\{\sum_j \mathbf{j}_n\cdot\hat{e}_j\mathbf{j}_e\cdot\hat{e}_j\frac{1}{k'^2-k^2}e^{i\mathbf{k}'\cdot(\mathbf{r}_n-\mathbf{r}_e)}+\mathbf{j}_n\cdot\hat{k}'\mathbf{j}_e\cdot\hat{k}'\frac{k'^2/k^2}{k'^2-k^2}e^{i\mathbf{k}'\cdot(\mathbf{r}_n-\mathbf{r}_e)}-\mathbf{j}_n\cdot\hat{k}'j_e\cdot\hat{k}'\frac{1}{k^2}e^{i\mathbf{k}'\cdot(\mathbf{r}_n-\mathbf{r}_e)}-\rho_n\rho_e\frac{1}{k'^2-k^2}e^{i\mathbf{k}'\cdot(\mathbf{r}_n-\mathbf{r}_e)}\Big\}.$$

The continuity equations

$$\nabla\cdot\mathbf{j}_n-ik\rho_n=0,\quad \nabla\cdot\mathbf{j}_e+ik\rho_e=0,$$

imply that the $(\mathbf{j}_n\cdot\hat{k}'\mathbf{j}_e\cdot\hat{k}')$ appearing in the second term above may, after two partial integrations, be replaced by $k^2\rho_n\rho_e$. This term is then seen to cancel the last term above, so that the scalar part is no longer explicitly present. The result is Eq. (7).

where it is understood that the singularity is defined in the usual way, $k^2 \to (k+i\epsilon)^2$.

The basic step in the conversion-correlation calculation involves the comparison of the probability of emission of a conversion electron in a given direction, $\hat{u}$, with the probability of emission of a corresponding gamma ray in the same direction. In the free-particle case considered, the final electron state is then to be taken as an eigenfunction of linear momentum, $\mathbf{p}=p\hat{u}$:

$$\psi_f(\mathbf{r}_e)=D(\tau_f,\mathbf{p})\, e^{i\mathbf{p}\cdot\mathbf{r}_e}. \tag{8}$$

The initial electron state is taken to be a $\kappa_i=-1$ state of zero momentum

$$\psi_i(\mathbf{r}_e)=D(\tau_i,\, \mathbf{p}=0)=\begin{pmatrix} 0 \\ \chi(\tau_i) \end{pmatrix}. \tag{9}$$

Here D is a Dirac spinor, and χ a Pauli spinor.

The integration over the electron coordinate in the interaction (7) may now be done immediately, and leads to the simple factor $\delta(\mathbf{k}'+\mathbf{p})$. The important point here is that the retardation denominator, $p^2-k^2=2k$, in the first term in (7), is small compared with the factor k^2 appearing in the denominator of the second term. If we now assume, in conformity with our previous neglect of penetration effects, that $\mathbf{j}_n\cdot\hat{e}$ is not especially small relative to $\mathbf{j}_n\cdot\hat{k}$, only the first term in (7) is important in the high-energy limit, and the second term is dropped.

The calculation of the spin-summed square of the interaction, which appears in the expression for the transition probability, is readily accomplished using the usual spin-trace methods. The result of these calculations is

$$\sum_{\tau_i,\tau_f} |H'|^2=\frac{4\pi^2}{k^2}\left(\frac{p^2}{k^2}\right)\sum_j \left|\int d\tau_n \mathbf{j}_n\cdot\hat{e}_j\, e^{-i\mathbf{p}\cdot\mathbf{r}_n}\right|^2. \tag{10}$$

In the high-energy limit, $p\to k$ and $\mathbf{p}\to\mathbf{k}$, where $\mathbf{k}=k\hat{u}$. The probability of conversion-electron ejection in the direction $\hat{u}$ is then proportional to

$$\frac{4\pi^2}{k^2}\sum_j \left|\int d\tau_n \mathbf{j}_n\cdot\hat{e}_j e^{-i\mathbf{k}\cdot\mathbf{r}_n}\right|^2. \tag{11}$$

This form is recognized to be identical to the usual one for gamma-ray emission in the same direction $\hat{u}$. In other words, in the case considered here, the probability for conversion and gamma-ray emission in any given direction become proportional. This proportionality is equivalent to the statement that the *conversion coefficient is independent of multipole order*, and, more importantly here

All particle parameters, b, $=+1$.

The above result then illustrates the correctness of the formalism (1) for this particular case of free particles in the high-energy limit. This proof is more general than the direct calculation discussed at the beginning of this section in that it includes all possible multipole mixtures, in addition to the particular case of ML, $E(L+1)$. The corresponding high-energy limit for the case of pure Coulomb wave functions is discussed below.

High-Energy Coulomb Limit

The radial integrals appearing in (2) and (3) can be evaluated analytically for the case of a pure Coulomb potential.[13] The limits of these integrals can then be explicitly evaluated as $k\to\infty$. It can be shown directly that this result is also obtained by evaluating the same radial integrals after replacing the final-state continuum wave functions and the Hankel functions by their asymptotic forms in the integrand, then performing the integration, and taking the high-energy limit of the result. This procedure is called the Casimir approximation in BR. The validity of the Casimir approximation has, therefore, been demonstrated by an explicit calculation.

In this high-energy limit we find the same results for the Coulomb case as obtained for the free-particle case[11]

$$\begin{aligned} T_m(L) &\to 1\,, \\ T_e(L+1) &\to -(L+2)\,, \\ \theta_e-\theta_m &\to \text{Phase of } T_e \to \pi\,. \end{aligned} \tag{12}$$

Substituting these results into the expressions (1) and those in Ref. 7, the same high-energy limits are obtained as before for the free-particle case; $b_\nu(ML)$, $b_\nu[E(L+1)]$, and $b_\nu[ML,\, E(L+1)]$ all approach $+1$.

The conclusion, that $b_\nu[ML,\, E(L+1)]\to +1$ in the high-energy limit of the Coulomb case, differs from the statement in BR concerning this limit by a sign reversal.[14]

One can also understand in a direct way the coincidence of the conversion-electron and gamma-ray correlation patterns in the high-energy limit of the Coulomb case. A derivation is given here which parallels the free-particle example.

This derivation starts with the conversion interaction written in the form of Eq. (7). In the high-energy limit, this interaction can again be approximated by its first, the transverse, term:

$$H'\cong -\frac{1}{2\pi^2}\sum_j \int d\mathbf{k}' \int d\tau_n \mathbf{j}_n\cdot\hat{e}_j\, e^{i\mathbf{k}'\cdot\mathbf{r}_n} \times\frac{1}{k'^2-(k+i\epsilon)^2}\int d\tau_e \mathbf{j}_e\cdot\hat{e}_j\, e^{-i\mathbf{k}'\cdot\mathbf{r}_e}. \tag{13}$$

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[13] M. E. Rose, G. H. Goertzel, B. I. Spinrad, J. Harr, and P. Strong, Phys. Rev. **83**, 79 (1951).

[14] The discussion of this high-energy limit in BR is further obscured by the need for another sign readjustment. This occurs in the expressions for the high-energy limits of the radial integrals labeled as $Q(\kappa,L,m)$ and $Q(\kappa,L,e)$ on p. 755 of BR. One or another of these expressions, but not both, must be multiplied by (-1).

The wave functions entering in the electron current, $\mathbf{j}_e=-e\psi_f^*\boldsymbol{\alpha}\psi_i$, are those appropriate for the Coulomb case: ψ_i is the initial bound-state wave function, and ψ_f is the final-state electron wave function.

The final state corresponds to an electron moving off with linear momentum $\mathbf{p}=p\hat{u}$. The wave function ψ_f is, then, that solution of the Dirac equation which is asymptotically a "plane-wave" plus an ingoing wave. It will be demonstrated below that, in the high-energy limit, only that region of the r_e integration is important for which this asymptotic form is a sufficient approximation. It will further be shown that in this limit the parts of ψ_f corresponding to ingoing waves contribute negligibly. For a potential which falls off more quickly than the Coulomb potential, $1/r$, these arguments immediately establish that ψ_f is to be replaced, in the high-energy limit, by a plane wave,

$$D(\tau,\mathbf{p})\,e^{i\mathbf{p}\cdot\mathbf{r}}.$$

The Coulomb case is somewhat more difficult on two counts: the well-known divergence in the forward direction, and the logarithmic phase. However, as will be seen, neither of these difficulties are actually important here.

The divergence difficulties of the Coulomb case have to do with the addition of an infinite series of partial waves. However, only a finite number of partial waves appear here; those picked out by the limited number of multipoles which are present in the nuclear radiation. As will be shown, these partial waves can be approximated by their asymptotic forms. In any case, the other partial waves enter with zero weight and can certainly be so approximated. The resultant series can then be immediately summed—not to a plane wave, but to a plane wave times a logarithmic phase factor. This last factor is related to the second difficulty noted above.

In the Coulomb case each partial wave has its phase shifted, not only by an r-independent amount, Δ, but also by an additional logarithmic term, $(\alpha Z\ln r)$. Since the phase factor, $\exp(i\alpha Z\ln r)$, is independent of the partial wave, it appears as a common factor in each of the outgoing wave components. Parenthetically, it can be noted that each ingoing wave appears with the factor $\exp(-i\alpha Z\ln r)$; however, it will be proven that ingoing waves contribute only negligibly, and so only the outgoing waves need be considered.

These conclusions are summarized by the statement that in the high-energy limit the final-state wave function can be approximated, to lowest order, by

$$e^{i\alpha Z\ln r}D(\tau,\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{r}}. \tag{14}$$

Two points remain to be proved: that the asymptotic form of the final-state wave function can be used in the whole region of integration, and that the ingoing wave parts of the final-state wave function contribute negligibly. The proofs are written below for the specific case of a $\kappa=-1$, relativistic $s_{1/2}$, initial electron state, since it presents the greatest difficulties. The proofs for other electron states and for weaker potentials are automatically included, although they could have been carried through with many fewer precautions.

To demonstrate the validity of the use of the asymptotic form of the wave function ψ_f in (13), the electron integral is divided into the regions $r_e<\xi$ and $r_e>\xi$, where ξ is a small distance to be defined more precisely in the course of the discussion. It will be shown that the contribution of the region $r_e<\xi$ is negligible, while in the region $r_e>\xi$ the asymptotic form is valid.

We begin with the contribution from the small region, $r_e<\xi$:

$$+\frac{1}{2\pi^2}\sum_j\int d\mathbf{k}'\frac{1}{k'^2-(k+i\epsilon)^2}\left[\int d\tau_n\mathbf{j}_n\cdot\hat{e}_j\,e^{i\mathbf{k}'\cdot\mathbf{r}_n}\right]\times\left[e\int_0^\xi d\tau_e\psi_f^*\boldsymbol{\alpha}\psi_i\cdot\hat{e}_j\,e^{-i\mathbf{k}'\cdot\mathbf{r}_e}\right]. \tag{15}$$

We are interested in the high-energy limit, that is, p large, and in particular, p large compared to the important momentum components of ψ_i, which are of order αZ. The distribution of $\mathbf{k}'$ determined by the last factor in (15) is then one lying about $-\mathbf{p}$ with a range of order $1/\xi$. The value of ξ is chosen so that this range, $1/\xi$, is small compared with p. In other words, the important range of the variable t, $\mathbf{t}\equiv\mathbf{k}'+\mathbf{p}$, is such that $t\ll p$. Therefore, on changing variables, noting that in the high-energy limit

$$k\gg t,\qquad p\gg t,$$
$$p-k\to E,\quad E=\text{total initial electron energy}, \tag{16}$$

and dropping terms of order $1/k$, we conclude that

$$\frac{1}{k'^2-(k+i\epsilon)^2}\to\frac{1}{2p(E-\hat{u}\cdot\mathbf{t}-i\epsilon)}. \tag{17}$$

Then, (15) becomes:

$$\frac{1}{2\pi^2}\frac{1}{2p}\sum_j\int d\mathbf{t}\frac{1}{E-\mathbf{t}\cdot\hat{u}-i\epsilon}\times\left[e\int_0^\xi d\tau_e\psi_f^*\boldsymbol{\alpha}\psi_i\cdot\hat{e}_j\,e^{i\mathbf{p}\cdot\mathbf{r}_e}\,e^{-i\mathbf{t}\cdot\mathbf{r}_e}\right]\times\left[\int d\tau_n\mathbf{j}_n\cdot\hat{e}_j\,e^{-i\mathbf{p}\cdot\mathbf{r}_e}\right]. \tag{18}$$

The nuclear integral in (18) has been simplified by noting that the retardation factor

$$e^{i\mathbf{k}'\cdot\mathbf{r}_n}=[e^{-i\mathbf{p}\cdot\mathbf{r}_n}\,e^{i\mathbf{t}\cdot\mathbf{r}_n}]\cong e^{-i\mathbf{p}\cdot\mathbf{r}_n}.$$

This follows from the fact that $r_n<R$, where R is the source radius, which $\to 0$ as $1/k$, in the high-energy limit. The integral over $\mathbf{t}$ is most easily carried out in rectangular coordinates, taking the z axis along $\hat{u}$

$$\int d\mathbf{t}\frac{1}{E-\mathbf{t}\cdot\hat{u}-i\epsilon}e^{-i\mathbf{t}\cdot\mathbf{r}_e}=-(2\pi)^3i\delta(x)\delta(y)e^{-iEz},\quad z>0$$
$$=0,\qquad z<0. \tag{19}$$

Inserting this $\mathbf{t}$ integration, (18) becomes

$$-\frac{(2\pi i)}{p}\sum_j\left[\int d\tau_n\mathbf{j}_n\cdot\hat{e}_j\,e^{-i\mathbf{p}\cdot\mathbf{r}_n}\right]\times\left[e\int_0^\xi dr\psi_f^*(r,0)\boldsymbol{\alpha}\psi_i(r,0)\cdot\hat{e}_je^{i(p-E)r}\right], \tag{20}$$

where $\psi(r,0)$ denotes the wave functions evaluated at $\theta=0,\phi=0$.

We can now obtain an upper bound for the possible value of (20). The initial function ψ_i is bounded by $a_ir^{\gamma_1-1}$, where a_i is, of course, independent of k. The function ψ_f is bounded by a quantity $[b_f(pr)^{\gamma_1-1}+c_f]$, where b_f, c_f are numbers independent of p or k, and where $\gamma_1\equiv(1-(\alpha Z)^2)^{1/2}$. To see this, it has to be noted (i) that in the high-energy limit ψ_f is a function of the combination (pr) only, (ii) that after the diverging $(pr)^{\gamma_1-1}$ is extracted, the remaining function is well behaved, with a maximum value which is, of course, independent of (pr), and (iii) that the normalization brings in no p-dependent factors. Therefore, the last factor in (20) is bounded by

$$a_i\left[b_f\frac{(p\xi)^{\gamma_1-1}}{2\gamma_1-1}+\frac{c_f}{\gamma_1}\right]\xi^{\gamma_1}. \tag{21}$$

We choose ξ so that $p\xi\to\infty$, while $\xi\to 0$ as $p\to\infty$; this is immediately realized by putting $\xi=1/p^n$, $0<n<1$. The bound, (21), then, approaches $(a_ic_f/\gamma_1)\xi^{\gamma_1}$. Therefore, the whole of term (20), which is the contribution from the small region $(r_e<\xi)$, vanishes as ξ^{γ_1} relative to the main term to be written explicitly later. In the region $r_e>\xi$, $pr_e>p\xi\to\infty$ in the high-energy limit, and so the asymptotic forms can be used. Finally, since the region 0 to ξ contributes negligibly, the modified plane-wave form of ψ_f can be used in the whole region $0\leq r_e\leq\infty$ in (13).

We now justify the neglect of the contributions of the ingoing wave components of ψ_f and those of the approximate form (14). That is, we compare the contribution to

$$\frac{1}{2\pi^2}\sum_j\int d\mathbf{k}'\frac{1}{k'^2-(k+i\epsilon)^2}\left[\int d\tau_n\mathbf{j}_n\cdot\hat{e}_j\,e^{i\mathbf{k}'\cdot\mathbf{r}_n}\right]\times\left[e\int_\xi^\infty d\tau_e\psi_f{}^*\boldsymbol{\alpha}\cdot\hat{e}_j\,e^{i\mathbf{k}'\cdot\mathbf{r}_e}\psi_i\right], \tag{22}$$

of the ingoing waves

$$\psi_f\to(e^{-ipr}/r)\,e^{\pm i\alpha Z\ln r}, \tag{23}$$

and of the outgoing wave

$$\psi_f\to(e^{+ipr}/r)\,e^{i\alpha Z\ln r}. \tag{24}$$

This comparison is most expeditiously handled by first carrying out the k' integration in (22). Since the electromagnetic propagator leads to outgoing electromagnetic waves, e^{+ikr}/r, the contribution from the ingoing electron wave is proportional to

$$\int_\xi^\infty dr\,r^2\left(\frac{e^{-ipr}}{r}\,e^{\pm i\alpha Z\ln r}\right)^*\frac{e^{ikr}}{r}r^{\gamma_1-1}\,e^{-\alpha Zr}\cong\frac{\Gamma(\gamma_1\mp i\alpha Z)}{[\alpha Z-i(k+p)]^{\gamma_1\mp i\alpha Z}}, \tag{25}$$

while that from the outgoing wave is proportional to

$$\int_\xi^\infty dr\,r^2\left(\frac{e^{+ipr}}{r}\,e^{i\alpha Z\ln r}\right)^*\frac{e^{ikr}}{r}r^{\gamma_1-1}\,e^{-\alpha Zr}\cong\frac{\Gamma(\gamma_1-i\alpha Z)}{[\alpha Z-i(k-p)]^{\gamma_1-i\alpha Z}}. \tag{26}$$

Since $(p-k)\to E$ in the high-energy limit, it is seen that the ingoing wave contribution vanishes relative to that of the outgoing wave as $\sim 1/k^{\gamma_1}$. This completes the demonstration of the validity of using (14) for ψ_f in the high-energy limit.

Having demonstrated the validity of the use of the modified plane waves, (14), in the high-energy limit, the probability of electron emission can now be calculated in a manner closely analogous to that given above for the free-particle case. From (13) and (14):

$$H'\cong+\frac{1}{2\pi^2}\sum_j\int d\mathbf{k}'\frac{1}{k'^2-(k+i\epsilon)^2}\left[\int d\tau_n\mathbf{j}_n\cdot\hat{e}_je^{i\mathbf{k}'\cdot\mathbf{r}_n}\right]\times\left[e\int d\tau_eD^*(\tau_f,\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{r}_e}e^{-i\alpha Z\ln r_e}e^{-i\mathbf{k}'\cdot\mathbf{r}_e}\boldsymbol{\alpha}\cdot\hat{e}_j\psi_i(\mathbf{r}_e)\right]. \tag{27}$$

It is now convenient to introduce the Fourier transform[15] of the combination $\exp(-i\alpha Z\ln r_e)\psi_i(\mathbf{r}_e)$,

$$e^{-\alpha Z\ln r_e}\psi_i(\mathbf{r}_e)=\int d\mathbf{q}e^{i\mathbf{q}\cdot\mathbf{r}_e}\phi(\mathbf{q}). \tag{28}$$

The integration over the electron coordinate leads to the factor $\delta(\mathbf{k}'+\mathbf{p}-\mathbf{q})$, and the trivial $\mathbf{k}'$ integration results, then, in the replacement of $\mathbf{k}'$ by $(\mathbf{q}-\mathbf{p})$. As in the derivation of (17), it follows that in the high-energy limit

$$\frac{1}{k'^2-(k+i\epsilon)^2}\to\frac{1}{2k}\,\frac{1}{E-\hat{u}\cdot\mathbf{q}-i\epsilon}, \tag{29}$$

and

$$e^{i\mathbf{k}'\cdot\mathbf{r}_n}\to e^{-i\mathbf{p}\cdot\mathbf{r}_n}, \tag{30}$$

so that

$$H'\to\frac{2\pi}{k}\sum_j\left[\int d\tau_n\mathbf{j}_n\cdot\hat{e}_j\cdot e^{-i\mathbf{p}\cdot\mathbf{r}_n}\right]\times\left[e\int\frac{d\mathbf{q}}{E-\hat{u}\cdot\mathbf{q}-i\epsilon}D^*(\tau_f,\mathbf{p})\boldsymbol{\alpha}\cdot\hat{e}_j\phi(\mathbf{q})\right]. \tag{31}$$

The probability of electron emission in the direction $\hat{u}$ is proportional to

$$\sum_{\tau_i,\tau_f}|H'|^2.$$

At first sight, it appears as if the second term in the denominator in (31) makes an angle-dependent contribution involving $\hat{u}$. However, after adding over all orientations of initial and final electron spin, there is no direction remaining with which $\hat{u}$ can couple. Furthermore, it can also be shown by direct calculation that there are no cross terms in the polarizations, $\hat{e}_j$, and that both polarizations are equally weighted. In

[15] The Fourier transform of the quantity $\exp(-i\alpha Z\ln r)\psi_i(\mathbf{r})$ is

$$\phi(\mathbf{q})=\begin{pmatrix}-i\psi(q)\chi_{+1}(\hat{q})\\ \theta(q)\chi_{-1}(\hat{q})\end{pmatrix}.$$

For the particular case of the K shell,

$$\theta(q)=+\frac{N_K}{2\pi^2}\frac{\Gamma(\gamma_1+1-i\alpha Z)}{2iq}\times\left\{\frac{1}{[\alpha Z-iq]^{\gamma_1+1-i\alpha Z}}-\frac{1}{[\alpha Z+iq]^{\gamma_1+1-i\alpha Z}}\right\},$$

and

$$\psi(q)=-\frac{N_K}{2\pi^2}\frac{\Gamma(\gamma_1+1-i\alpha Z)}{2iq}\times\frac{i}{1+\gamma_1}\left[(\gamma_1+1+i\alpha Z)\frac{\alpha Z}{q}\left(\frac{1}{[\alpha Z-iq]^{\gamma_1-i\alpha Z}}-\frac{1}{[\alpha Z+iq]^{\gamma_1-i\alpha Z}}\right)-\alpha Z\frac{\alpha Z}{q}\left(\frac{1}{[\alpha Z-iq]^{\gamma_1+1-i\alpha Z}}-\frac{1}{[\alpha Z+iq]^{\gamma_1+1-i\alpha Z}}\right)\right],$$

where

$$N_K=\frac{(1+\gamma_1)^{1/2}(2\alpha Z)^{\gamma_1+1/2}}{[2\Gamma(2\gamma_1+1)]^{1/2}}.$$

other words, in the high-energy limit one may write

$$\sum_{\tau_i,\tau_f}|H'|^2=\frac{4\pi^2}{k^2}\sum_j\left|\int d\tau_n \mathbf{j}_n\cdot\hat{e}_j\, e^{-i\mathbf{p}\cdot\mathbf{r}_n}\right|^2$$

$$\times\sum_{\tau_i,\tau_f}\left|e\int d\mathbf{q}\frac{1}{E-\hat{\sigma}\cdot\mathbf{q}-i\epsilon}D^*(\tau_f,p\hat{\sigma})\boldsymbol{\alpha}\cdot\hat{\tau}\phi(\mathbf{q})\right|, \quad (32)$$

where $\hat{\sigma}$, $\hat{\tau}$ are two arbitrary, mutually perpendicular, unit vectors. This expression (32) immediately demonstrates that the probability of electron emission in the high-energy limit is precisely proportional to that for gamma-ray emission without any further angular dependences. This proportionality is equivalent to the state that, in this high-energy limit, the *conversion coefficient is independent of multipole order*, and that:

All particle parameters $b=+1$.

The actual value of the high-energy limit of the internal-conversion coefficient can be obtained by evaluating the second, the electron, factor in Eq. (32). This is carried out most easily by going back into configuration space through

$$\phi(\mathbf{q})=\frac{1}{(2\pi)^3}\int d\tau_e e^{-i\mathbf{q}\cdot\mathbf{r}_e}e^{-i\alpha Z\ln r_e}\psi_i(\mathbf{r}_e), \quad (33)$$

taking $\hat{\sigma}$ along the z axis, and carrying out the $\mathbf{q}$ integration first. The result for conversion in the shell, i, is

$$\frac{(j_i+\frac{1}{2})}{2k}\left|e\int dr(g_i+if_i)e^{-i\alpha Z\ln r-iEr}\right|^2. \quad (34)$$

For the particular case of the K shell this becomes

$$\left[\frac{(2\alpha Z)^{2\gamma_1+1}}{2}\right]\frac{\alpha}{k}\frac{|\Gamma(\gamma_1+i\alpha Z)|^2}{\Gamma(2\gamma_1+1)}e^{-2\alpha Z\arcsin\gamma_1}. \quad (35)$$

EXPERIMENTAL CONSEQUENCES

There are many directional-correlation experiments involving the conversion electrons of mixed-multipole transitions reported in the literature. To the extent that the results of these papers depend on the use of BR Table IV for the interference particle parameters, they should be reanalyzed. Here, we consider two experimental cases of special interest: Tl^{203} and Tl^{201}, which were originally analyzed using Eq. (101) together with Table IV of BR.[16,17] A consistent interpretation of these experiments could only be obtained by the introduction of large penetration[10] effects for the l-forbidden $\frac{3}{2}\rightarrow\frac{1}{2}$ transition. (Large penetration effects correspond to a "large-λ" solution, where λ is the ratio of the penetration and gamma-ray matrix elements.) However, it was pointed out by Deutch and by Gerholm (private communications), that a simple sign reversal of the mixture term might allow a consistent interpretation of the data in terms of small penetration effects ("small λ").

The data of Gerholm, Petterson, Van Nooijen, and Grabowski[17] on the Tl^{203} case have been reanalyzed using the new sign for the interference effect. The reported experimental results consist of gamma-gamma, electron-gamma, and gamma-electron correlation coefficients of the 400 kV–279 keV ($\frac{5}{2}+\rightarrow\frac{3}{2}+\rightarrow\frac{1}{2}+$) cascade. The measured gamma-ray and conversion-electron correlations of the 400-keV transitions are consistent with each other over a large range of the multipole mixing parameter, $\delta(279)$, within the experimental errors. Analysis of the data for the retarded 279-keV transition has also been performed. The particle parameters are those used by Gerholm *et al.*, except for the sign change in $b_2[M1,E2]$. Combining the correlation results with the K-conversion coefficient data of Herrlander and Graham[18] shows that the experimental results are still consistent with a large value of λ. However, the data are also consistent, within about one and a half standard deviations, with a small λ value; the degree of agreement is improved if the electron-gamma correlation coefficient is made slightly larger than that reported.

The earlier results of Deutch and Goldberg[16] for Tl^{203} have also been reanalyzed, using the new values of $b_2[M1, E2]$. The correlation results are found to be consistent, within the rather large quoted errors, with the work of Herrlander and Graham,[18] for both large and small values of λ.

The data of Pettersson, Gerholm, Grabowski, and Van Nooijen[17] on Tl^{201} have been similarly reanalyzed. It does not seem possible to obtain a consistent interpretation of all the measured correlations and conversion coefficients to within the quoted experimental error.

The other experimental cases have not yet been reanalyzed.[19]

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[19] *Note added in proof.* J. S. Geiger has sent us a preprint of a paper [Phys. Letters **7**, 48 (1963)] on experimental electron correlations in I^{127}. The results suggested to him that the sign of $b[M1,E2]$ as given in BR should be changed. He also pointed out that this change permitted the existing Tl correlation data to be interpreted in terms of small λ values. We thank Dr. Geiger for his early communication of these results.